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Some Contributions to the Geometry of Plane Transformations.

By Tobias Dantzig.

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§ 1. *The Indicatrix of a Transformation.*

Let T be a continuous plane point-to-point transformation sending a plane Π into itself. To fix the idea I shall further assume that T transforms every point P in the plane Π into a unique point $\overline{P}$. The latter I shall call the image of P by T. In the concluding pages of this paper is to be found an extension of most of the considerations here developed to the case of a $p:q$ transformation.

Let P be a point of the plane Π and let $\overline{P}$ be it's image by T. Consider an arbitrary curve C passing through P and let t be its tangent at the point P. The curve C is transformed by T into a curve $\overline{C}$ which will pass through $\overline{P}$ and have there a unique tangent $\bar{t}$, providing the point P is a simple point of the curve C. Let the two lines t and $\bar{t}$ meet in a point τ. Then it is easy to see that,

The position of the point τ is independent of the curve C and is perfectly determined once the points P and $\overline{P}$ and the line t are given.

Indeed let C' be any other curve passing through P and having there t for tangent. The image $\overline{C}'$ will pass through $\overline{P}$ and, *since T conserves contact*, will touch at $\overline{P}$ the line $\bar{t}$. The point τ I shall call the *tactal* of the line t at the point P.

Assuming now the points P and $\overline{P}$ fixed, to each line t through P will correspond a unique line $\bar{t}$ through $\overline{P}$. When the line t turns about P a *one-to-one correspondence* is established between the two pencils having their vertices in P and $\overline{P}$, respectively, and the point is the *product* of two corresponding rays. It follows at once, therefore, that

The locus of the tactal τ for one and the same point P, is a conic K passing through P and $\overline{P}$.

The conic K is uniquely determined for every point P of the plane Π. I shall call it the *indicatrix of the transformation T for the point P.* The totality of all the indicatrices of the plane form a system of conics depending on two parameters.

When the points P and $\overline{P}$ are known, the determination of K requires the knowledge of *three* tactals. These can be obtained by tracing through P three arbitrary curves having different tangents at P. The proper selection of these curves will facilitate the construction of K. The following obvious remarks may help to guide in this selection.

The line $P\overline{P}$ I shall call the *bridge* of the transformation at P. It follows at once that

If a curve C touches the bridge $P\overline{P}$ at P, its image $\overline{C}$ will touch at $\overline{P}$ the indicatrix K, and, conversely, if C touches K at P, $\overline{C}$ will touch $P\overline{P}$ at $\overline{P}$.

If C and $\overline{C}$ have parallel tangents at corresponding points P and $\overline{P}$, the direction of these tangents is an asymptotic direction of the indicatrix K for P.

The importance of the introduction of the indicatrix can be best illustrated by the following problem:

Determine a geometrical construction of the tangent $\overline{t}$ at a point $\overline{P}$ of a plane curve $\overline{C}$, knowing that $\overline{C}$ is the image of a curve C by a transformation T.

Assume that the geometrical construction of the tangent t of the curve C is known, and let K be the indicatrix of T for the corresponding point P. If t meets K in a second point τ, τ is evidently the tactal of t at P, and consequently $\tau\overline{P}$ is the tangent sought. It is interesting to remark that if K is determined by five points (P, $\overline{P}$ and three tactals τ_1, τ_2, τ_3) and t is known, the *construction can be completed by projective means only* (Pascal's theorem).

These considerations bring out the following property: The indicatrix of a transformation T for a point P can serve as a *first asymptotic element of T at P, characterizing the behavior of T in the vicinity of the point P,* to the same degree as the tangent at a point of a plane curve characterizes the behavior of the curve at the point.

§ 2. *Equation of Indicatrix.*

To obtain the general equation of the indicatrix assume that the equations of the transformations T are given in the explicit form

$$\overline{x}=\phi(x, y), \quad \overline{y}=\psi(x, y), \tag{1}$$

where ϕ and ψ are single-valued continuous functions of x and y, admitting first derivatives which are also single valued and continuous; and where $x, y;$

$\bar{x}$, $\bar{y}$ are the coordinates of P and $\bar{P}$, respectively, in a system of cartesian coordinates. We shall set

$$\frac{\partial\phi}{\partial x}=p,\quad \frac{\partial\phi}{\partial y}=q,\quad \frac{\partial\psi}{\partial x}=p',\quad \frac{\partial\psi}{\partial y}=q'.$$

The equations of the corresponding tangents t and $\bar{t}$ at P and $\bar{P}$ being evidently

$$\frac{X-x}{dx}=\frac{Y-y}{dy}, \text{ and } \frac{X-\bar{x}}{d\bar{x}}=\frac{Y-\bar{y}}{d\bar{y}}, \tag{2}$$

we are to eliminate the differentials from these equations with the aid of the identical relations

$$d\bar{x}=pdx+qdy,\quad d\bar{y}=p'dx+q'dy. \tag{3}$$

We have, by setting the common ratios in (2) equal to $1/\rho$ and $1/\bar{\rho}$, respectively, the two equations

$$\bar{\rho}(X-\bar{x})=[p(X-x)+q(Y-y)]\rho,\quad \bar{\rho}(Y-\bar{y})=[p'(X-x)+q(Y-y)]\rho,$$

and eliminating ρ and $\bar{\rho}$ we obtain the equation of the indicatrix in the form

$$\frac{p(X-x)+q(Y-y)}{X-\bar{x}}=\frac{p'(X-x)+q'(Y-y)}{Y-\bar{y}}. \tag{4}$$

In this form it is apparent that the indicatrix is circumscribed to the quadrilateral whose four sides are $X=\bar{x}$, $Y=\bar{y}$, $p(X-x)+q(Y-y)=0$, $p'(X-x)+q'(Y-y)=0$. Collecting the terms we have

$$p'(X-x)(X-\bar{x})-p(X-x)(Y-\bar{y})-q'(X-\bar{x})(Y-y)$$
$$-q(Y-y)(Y-\bar{y})=0. \tag{5}$$

The character of the conic K is determined by the discriminant of the quadratic part of equation (5)

$$\Delta(x,y)=(p-q')^2+4p'q. \tag{6}$$

Assuming now that the transformation T is real, the curve $\Delta(x,y)=0$ divides the plane Π into two *regions: E* and *H*. In the region E, $\Delta<0$ and the indicatrix K is an *ellipse*, in the region H, $\Delta>0$ and the conic K is a *hyperbola*, on the *discriminant curve* itself the conic is a parabola. I shall call these regions the *elliptic* and *hyperbolic regions* respectively. If the transformation admits the whole plane for an *elliptic region* I shall call it an *elliptic transformation*, and the terms *hyperbolic* and *parabolic* transformations should be taken in the same sense.

From the consideration of the preceding section it follows at once that

At any point P in the region H there exist two rays α and β, such that, if a curve C passing through P admits one of the rays for tangent, its image $\overline{C}$ will have a parallel tangent at $\overline{P}$. These rays α and β are the asymptotic directions of the indicatrix K. They are coincident at every point of the discriminant curve Δ and become imaginary in the region E.

Of course, the locus $\Delta=0$ is not necessarily *singly connected.* Consider for instance the *transcendental* transformation

$$\bar{x}=\cos y, \quad \bar{y}=\sin x.$$

We have here

$$\Delta=-4\cos x\cdot\sin y,$$

and the discriminant locus consists of the lines

$$x=\pi/2+k\pi \text{ and } y=l\pi.$$

The locus divides the plane into an *infinite number of squares* whose sides are parallel to the coordinate axes and which alternately constitute the *elliptic* and *hyperbolic regions.*

§ 3. *The Invariant Points.*

The *invariant* points of a transformation T given by equations (1) are obtained by setting

$$\bar{x}-x=\xi=0, \quad \bar{y}-y=\eta=0. \tag{7}$$

If equations (7) admit but a discrete number of simultaneous solutions, the invariant points are isolated. It may happen, however, that the equations $\xi=0$, $\eta=0$ have a factor $f(x, y)$ in common. The curve $f=0$ is then a *curve of fixed points.*

I shall prove now that:

If S is an invariant point the indicatrix K for S degenerates into two straight lines passing through S.

Indeed, the image $\overline{C}$ of an arbitrary curve C, passing through S must also go through S. If then the tangents t and $\bar{t}$ are distinct, the tactal of t is in S, if they coincide, the tactal is indetermined on t. The coincidence of t and t will occur only when the tangent is an asymptotic direction of K. Consequently the *conic K will degenerate at an invariant point into its asymptotes.*

According as the invariant point is situated in the elliptic or hyperbolic regions, or on the discriminant curve, we shall call the invariant points *elliptic, hyperbolic, or parabolic.*

An elliptic invariant point is necessarily isolated, if real.

Indeed the indicatrix degenerates into two imaginary lines at such a point. If there were another real invariant point S', in the vicinity of S, and at an infinitesimal distance from it, SS' would necessarily be an asymptotic direction and the latter would be real, contrary to the hypothesis.

As a corollary we see at once *that in the region E no real curve can touch its image at an invariant point.*

If then T admits of a curve of fixed points the latter must be entirely in the region H or make part of the discriminant curve $\Delta=0$. *In either case the curve of fixed points is the envelope of one of the branches of the indicatrix K.*

Conversely, if the indicatrix of a point P degenerates into two lines passing through P, the point P is necessarily an invariant point.

EXAMPLE: Consider the transformation defined by the equations

$$\bar{x}=(x^2+y^2+2ax-a^2)/2a, \quad \bar{y}=(x^2+y^2+2ay-a^2)/2a.$$

We have in this case

$$\Delta=[(x+y)/a]^2.$$

The discriminant curve is the line $x+y=0$ counted double. The transformation is hyperbolic throughout the plane, except on Δ, where the indicatrix is a parabola. We have here

$$\xi=\eta=(x^2+y^2-a^2)/2a,$$

and the transformation admits of a fixed circle

$$x^2+y^2=a^2.$$

At any point of this circle the indicatrix degenerates into the tangent to the circle and the line parallel to $x=y$.

§ 4. *The Jacobian.*

The degeneracy of the indicatrix at an invariant point leads us to examine the general case of degeneracy of the conic K. For this purpose make in (5) the substitution

$$\xi=\bar{x}-x, \quad \eta=\bar{y}-y.$$

The equation of K then becomes

$$p'(X-x)^2-(p-q')(X-x)(Y-y)-q(Y-y)^2 \\ -(p'\xi-p\eta)(X-x)-(q'\xi-q\eta)(Y-y)=0. \quad (8)$$

The condition that the conic degenerates is

$$(pq'-qp')[p'\xi^2-(p-q')\xi\eta-q\eta^2]=0. \quad (9)$$

The degeneracy locus therefore consists of the two curves

$$J(x,\, y)=pq'-qp'=0, \tag{10}$$

and

$$D(x,\, y)=p'\xi^2-(p-q')\xi\eta-q\eta^2=0. \tag{11}$$

The first is the *Jacobian of the transformation*, the second I shall call *the degeneracy locus proper.* We see that the invariant points make part of the locus $D=0$.

Geometrically the two cases can be characterized in the following manner. If P is not an invariant point it may happen that the degenerate conic has one branch α passing through P, the other β passing through $\overline{P}$. The tactal of any line t is evidently on β; β is therefore the line corresponding to any direction t through P. The transformation therefore establishes a pseudo-correspondence between the two pencils. Such points are characterized by the fact that the image of an arbitrary curve passing through them has a fixed tangent β, independent of the direction of the tangent t. If m and $\overline{m}$ are the slopes of two curves C and C at corresponding points we have in general

$$\overline{m}=\frac{d\overline{y}}{d\overline{x}}=\frac{p'+q'm}{p+qm}\,, \tag{12}$$

and the condition that this homographic correspondence degenerates into a pseudo-correspondence is

$$J=0.$$

At any point of the Jacobian of the transformation T the indicatrix degenerates into two lines of which but one α passes through P, the other β containing the point $\overline{P}$. Conversely, if the degeneracy is of this type, the point P is on the Jacobian.

It follows from these considerations that *if a curve C has in a point P belonging to the Jacobian of T a double point, its image $\overline{C}$ will have in $\overline{P}$ a cusp, and the cuspidal tangent is the branch β of the indicatrix.*

§ 5. *The Degeneracy Locus Proper.*

We shall now interpret the equation (11)

$$D=p'\xi^2-(\gamma-q')\xi\eta-q\eta^2=0.$$

The equation of the indicatrix in this case is

$$p'(X-x)^2-(\gamma-q')(X-x)(Y-y)-q(Y-y)^2=0, \tag{13}$$

and the condition $D=0$ expresses that one branch of the degenerate conic coincides with the bridge

$$(X-x)/\xi=(Y-y)/\eta. \tag{14}$$

At every point of the locus $D=0$ the indicatrix degenerates into two lines of which one α coincides with the bridge at P. *If the bridge is a tangent to any curve C passing through P, it is also tangent to the transformed curve C at P. Conversely, if the degeneracy is of this type, the point P at which it occurs belongs to the locus $D=0$.*

If we set $\omega=\eta/\xi$ and take in consideration that

$$\frac{\partial\xi}{\partial x}=p-1,\quad \frac{\partial\xi}{\partial y}=q,\quad \frac{\partial\eta}{\partial x}=p',\quad \frac{\partial\eta}{\partial y}=q'-1,$$

equation (11) takes the remarkable form

$$\frac{\partial\omega}{\partial x}+\omega\frac{\partial\omega}{\partial y}=0. \tag{15}$$

Consider the curves, $\omega(x, y)=\text{const.}$ Through any point in the plane passes a unique curve of this family. The transformation T *shifts every point of the curve, $\omega=const.$, in a constant direction whose slope is ω.* The slope of the tangent at any point of the curve is equal to

$$m=-\frac{\partial\omega}{\partial x}\Big/\frac{\partial\omega}{\partial y}.$$

Relation (15) expresses that at a point where any of the curves $\omega=\text{const.}$ crosses the degeneracy locus D *the curve touches the bridge of the point.*

Consider in particular a line l which the transformation T leaves invariant. At every point P of such a line the indicatrix degenerates into the line l itself and a second line which in general crosses l in a point different from P or $\bar{P}$. It follows therefore that

An invariant straight line of a transformation necessarily makes part of the degeneracy locus $D=0$. At all points of such a line the indicatrix degenerates and one branch of it is the line l itself.

From the discussion of Section 2 we conclude that *a real invariant line can not penetrate into the elliptic region of the plane*, for the degenerate indicatrix is necessarily *real* at all points of such a line.

The same remark holds for any continuous branch of the degeneracy locus and the Jacobian: *they can have in the elliptic region of the plane but isolated points.*

As a corollary it follows that an elliptic transformation admits of no invariant lines unless *a part of its discriminant locus be an invariant straight line.*

§ 6. *Directed Transformations.*

The case when the indicatrix degenerates at any point in the plane deserves special attention. Equation (9) shows at once that this will happen: *First*, when we have identically

$$\xi=0, \quad \eta=0.$$

In this case we have $x=x$, $y=y$, and T is the *identical transformation. Second*, when J is identically zero. There exists then a functional relation between x and y, and T is a *pseudo-transformation.* Equation (1) is but a *parametrical representation* of a curve, or we can regard T as transforming the plane Π into a *one-dimensional configuration.* Leaving these cases which do not present any interest, we still have to interpret the identical vanishing of D. The general integral of equation (15) is easily found to be

$$y-\omega x=F(\omega), \tag{16}$$

where F is an arbitrary function and ω is the slope of the bridge of any point. Since, on the other hand, the equation of the bridge at a point $P(x, y)$ is

$$Y-\omega X=y-\omega x;$$

equation (16) expresses that the totality of the bridges in the plane depend on but one parameter. The bridges therefore envelope a curve W. We shall call this type of transformation a *directed transformation,* and the curve W *the directrix of T.*

If the directrix is an algebraic curve of *class k*, the multiplicity of such a transformation is at least k to k. The determination of such a transformation therefore requires the knowledge of the directrix and the *mode of correspondence* between the points P and $\overline{P}$ on the bridge. If the latter is a p-to-q correspondence, the multiplicity of the transformation is evidently $kp:kq$.

Assume, in particular, that the correspondence between the points on the bridge is *one-to-one.* All the *singular* points of the transformation are evidently situated either on the directrix or on one of the double tangents to the directrix. The transformation is *regular* at any other point of the plane, and any one of its branches may be regarded as a *one-to-one* transformation. *At any such regular point the indicatrix degenerates into the bridge and a second line β.**

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* Concerning the meaning of singular and regular points see Section 21.

§ 7. *Central Transformation.*

If on a directed transformation we impose the additional condition of being of *multiplicity* 1:1, the directrix W must reduce to a *point* 0. The transformation is characterized by the fact that all bridges concur in a point, the center of the transformation.

The indicatrix of such a *central* transformation degenerates at any point in the plane into the ray of the point and a second line β.

From the preceding section it is seen that *the center is a singular point.* If we take it for origin, the equation of any central transformation can be put in the form

$$\bar{x}=g(x, y) \cdot x, \quad \bar{y}=g(x, y) \cdot y. \tag{17}$$

The curves $g(x, y)=$const. are of special interest. We have indeed,

$$\xi=(g-1) \cdot x, \quad \eta=(g-1) \cdot y, \tag{18}$$

and the locus $g=1$ is a curve of fixed points.* We find, on the other hand,

$$J=g(xg_x+yg_y+g),$$

and the *Jacobian* consists of the curves $g=0$ and $xg_x+yg_y+g=0$.

We find for the discriminant

$$\Delta=(xg_x+yg_y)^2, \tag{19}$$

and the two branches have for equation

$$yX-xY=0, \quad g_xX+g_yY=g_xx+g_yy-g(g-1). \tag{20}$$

Consequently *the second branch of the indicatrix is parallel to the tangent of the curve $g=$const. passing through the point.* At any point of the discriminant curve the curve $g=$const. touches at P the radius OP. From this it follows that *the condition that a central transformation be parabolic is that the family of curves $g=$*const. *consists of the pencil of lines through O.* Analytically the condition can be obtained by integrating (19), which gives

$$\bar{x}=g(y/x) \cdot x, \quad \bar{y}=g(y/x) \cdot y. \tag{21}$$

If we put this in polar coordinates we arrive at the conclusion that the transformation

$$\bar{\Theta}=\Theta, \quad \bar{\rho}=g(\theta)\rho \tag{21'}$$

is *parabolic throughout the plane.*

The curve $g(x, y)=k$ and its image are evidently *homothetic* with respect to the origin, the *ratio of homothety* being k.

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* It is assumed here that if the function $g(x, y)$ has a denominator, this latter does not contain x or y as a factor.

§ 8. *The Foci of a Transformation.*

Whenever a transformation admits of an *invariant pencil* of straight lines I shall call the vertex of such a pencil *a focus* of the transformation. The following theorem brings out the importance of these points.

All indicatrices of a transformation pass through its foci, and conversely, if all the indicatrices of a transformation pass through a fixed point, the pencil of lines through this point is left invariant.

For let K be the indicatrix for any point P, $\overline{P}$ the image of P, and C a focus of the transformation. By hypothesis PC is transformed into $\overline{P}C$, C is therefore a tactal. Conversely, if all the indicatrices pass through a point C, consider any line l through C and the family of indicatrices relative to the points on l. To each point P corresponds on the conic K a point $\overline{P}$ such that $\overline{P}C$ is the tangent at $\overline{P}$ to the image of l. All the tangents of $\overline{l}$ concur therefore in C, which proves that $\overline{l}$ is a straight line passing through C.

The transformation being assumed one-to-one, there exists a *homographic correspondence* between the two *superposed pencils* l and $\overline{l}$. We see therefore, that in any invariant pencil there are *two invariant lines* u and v, the *double rays* of the pencil through C. I shall call the invariant pencil *elliptic*, *hyperbolic*, or *parabolic*, according as u and v are congugate imaginary, real, or coincident.

If an invariant pencil contains more than two invariant lines, every line in the pencil is invariant. The homographic correspondence between the two pencils l and $\overline{l}$ reduces to *identity* and the transformation is a *central* transformation.

I shall say that a transformation is *unifocal*, *bifocal*, etc., if it admits of one, two, etc., foci. It follows from the considerations developed that *among the indicatrices of a unifocal transformation there are two families of degenerate conics.* All the indicatrices of any one of the two families have an invariant line of the transformation for one common branch.

No one-to-one transformation can possess more than three non-collinear foci.

For, assume that there are four foci, C_1, C_2, C_3, C_4, no three of which are on a straight line. All indicatrices will have to pass through the four foci. The system of indicatrices in the plane would reduce to a *pencil of conics* and any one of the conics would have to serve as indicatrix for any point on it, which is absurd.

We shall see later that if a transformation admits of *three non-collinear foci* it is a *collineation.*

If three of the foci of a transformation are collinear, any point on the line δ *containing these foci is also a focus.*

Indeed, since any indicatrix must pass through all the foci, they will all have the *axis* δ for common branch, and, consequently, contain any point on δ. We shall see later that the transformation is a *perspective.*

§ 9. *Unifocal Transformation.*

If the focus of a unifocal transformation be taken for origin, one of the equations of the transformation can be readily obtained by expressing that there exists between y/x and $\overline{y}/\overline{x}$ a *homographic relation*

$$Ax\overline{y}+By\overline{y}=Cx\overline{x}+Dy\overline{x}. \tag{22}$$

By setting

$$\overline{x}/(Ax+By)=\overline{y}/(Cx+Dy)=g(x, y),$$

the equations of the transformation are put into the form

$$\overline{x}=(Ax+By)\cdot g(x,y), \quad \overline{y}=(Cx+Dy)\cdot g(x,y); \tag{23}$$

and this is the resultant of the two transformations

$$\overline{x}=x'g(x',y'), \quad \overline{y}=y'g(x',y') \quad \text{and} \quad x'=Ax+By, \quad y'=Cx+Dy.$$

Consequently,

Any unifocal one-to-one transformation is the product of a central and a linear transformation.

Let now T_1 and T_2 be two one-to-one transformations, one sending a point P into P_1, the other sending P_1 into P_2, and let $T_1T_2=T_3$ represent the product of the two transformations. Let, moreover, K_1 be the indicatrix of T_1 for P, K_2 that of T_2 for P_1, and K_3 that of T_3 for P. *The conic K_3 must go through the three intersections of K_1 and K_2 others than P_1.* Indeed, if Q be one of these intersections, QP and QP_1 are corresponding rays in the pencils (P) and (P_1). Since, on the other hand, P_2 is on K_2, QP_1 and QP_2 are corresponding rays in the pencils (P_1) and (P_2). It follows therefore that QP and QP_2 must be corresponding rays in the pencils (P) and (P_2) as determined by the transformation T_3. Q is therefore on the conic K_3.

This lemma permits the determination of the indicatrix of a product of two transformations when the indicatrices of the components are known. For it gives five points of the conic K_3: P, P_2, Q, Q_1, Q_2. In particular it can be applied to any unifocal transformation thus *reducing the determination of the indicatrix to that of a central transformation.*

§ 10. *Bifocal Transformation.*

Consider a one-to-one transformation T admitting *two* foci, C_1 and C_2. We shall call the line $l=C_1C_2$ the *line of foci.* Each of the pencils C contains *two invariant lines.* Denote these by u_1, v_1 and u_2, v_2, respectively. It is clear that the transformation admits of *four invariant points,* u_1u_2, u_1v_2, v_1u_2, v_1v_2. The foci are singular points; the image of C_1 is indetermined on the line l_2 corresponding to l in the pencil (C_2), and similarly C_2 is transformed into the line l_1 corresponding to l in the first pencil. The transformation admits of a third singular point G, the intersection of the lines l'_1 and l'_2 to which l corresponds in the pencils (C_1) and (C_2), respectively. The image of G is therefore the line of foci.

On the other hand T can be considered in two ways as a *unifocal transformation.* It follows that the equations of the transformation are of the form

$$A_1\bar{x}+B_1\bar{y}+C_1=0, \quad A_2\bar{x}+B_2\bar{y}+C_2=0, \tag{24}$$

where the coefficients are linear expressions in x and y.

T is therefore a "Magnus transformation."* Any straight line is transformed into a conic going through *three fundamental points*, which in this case are C_1, C_2 and the intersection F of l'_1 and l'_2. Conversely, any conic through C_1, C_2 and the intersection G of l'_1 and l'_2 is transformed into a straight line. In particular, the pencil of lines through G goes into the pencil through F.

Let L, M, N be the fundamental points of any Magnus transformation T, and L', M', N' the fundamental points of the *inverse* transformation T^{-1}, and consider T as operating between two superposed planes Π and Π'. Then a linear transformation could bring two pairs of corresponding points in coincidence, say L' into L and M' into M. Under these circumstances the pencils (L') and (M') would be superposed with (L) and (M), and the points L and M would be foci of the new transformation. Therefore

The most general Magnus transformation can be considered as the product of a linear transformation and a bifocal transformation.

§ 11. *Conformal Transformations, Elliptic Type.*

If a transformation conserves angles both in size and in sense of rotation I shall call it an *elliptic* transformation, otherwise a *hyperbolic conformal transformation.*

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* See Magnus, "Nouvelle méthode," etc. *Crelle,* 8 pp. 52–63.

An elliptic conformal transformation has for indicatrix at any point of the plane a circle.

The theorem is evident geometrically. Indeed, let t and t' be any two rays of the pencil (P), and let $\bar{t}$ and $\bar{t}'$ be the corresponding two rays in the pencil $(\bar{P})$. By hypothesis the angles (t, t') and $(\bar{t}, \bar{t}')$ are equal and have the same sense of rotation; the locus of the point τ is therefore a circle.

Conversely, if the indicatrix is a circle, the angles are conserved. Analytically the condition that equation (8) represent a circle is given by

$$p = q', \quad q = -p', \tag{25}$$

and this is the well-known condition that a transformation be "*directly*" *conformal.*

A conformal transformation of this type is necessarily *elliptic* throughout the plane, and the discriminant curve is imaginary. *An invariant point of an elliptic conformal transformation is necessarily isolated and admits for indicatrix the isotropic lines passing through it.*

There are no real invariant lines.

It follows from the developments of the preceding section that *a conformal one-to-one transformation of this type is a bifocal transformation having two elliptic foci in the cyclic points at infinity I and J.*

All straight lines in the plane are transformed into conics going through the foci, *i. e.*, into *circles*, and these circles will have a third fixed point in common, real or imaginary. *The lines of the plane have for images circles having a radical center in common.* This radical center is *real*, for the lines l and l, to which the line at ∞ corresponds in the pencils I and J, respectively, are conjugate imaginary. For the same reason the *four invariant points of a conformal transformation are real.*

Let F be the radical center of the circles, and let G be the radical center of the circles which are transformed into lines. Then from the preceding section it is seen that F is transformed into the line at infinity and G is the image of the same line, *i. e.*, F and G are *vanishing points.*

§ 12. *Conformal Transformations, Hyperbolic Type.*

The case of the *hyperbolic conformal* transformation, in which the sense of rotation of angles is reversed, is connected with the preceding case by the following theorem:

Any hyperbolic conformal transformation is the product of a circular transformation and reflexion.

Indeed, let

$$\bar{x}=\phi(x, y), \quad \bar{y}=\psi(x, y)$$

be the equations of a circular transformation. If we make the substitution

$$x=x', \quad y=-y',$$

we will have

$$\frac{\partial\bar{x}}{\partial x'}=p, \quad \frac{\partial\bar{x}}{\partial y'}=-q, \quad \frac{\partial\bar{y}}{\partial x'}=p', \quad \frac{\partial\bar{y}}{\partial y'}=-q',$$

the condition $p=q'$, $p'=-q$ is therefore transformed into

$$p=-q', \quad p'=-q. \tag{26}$$

Nevertheless there are some advantages in treating the problem directly. The following theorem is true:

The indicatrix of a conformal transformation of the hyperbolic type is an equilateral hyperbola admitting the bridge for diameter.

The proof can be based on this classical property of an equilateral hyperbola: Any arc of the curve is viewed at equal or supplementary angles from the extremities of any diameter of the curve. It can also be regarded as the geometrical interpretation of conditions (26) in the equation of the indicatrix (8).

Let us apply these considerations to the determination of all *central conformal transformations.* It is clear that, if real, these can not be of the circular type as the indicatrix degenerates. Therefore the indicatrix is a degenerate equilateral hyperbola admitting the bridge for diameter. The second branch of the indicatrix is then the *perpendicular bisector* to $P\bar{P}$. The curves $g=$const. are *orthogonal trajectories* to the bridges as well as their images, that is, *circles having O for center.* The transformation is necessarily *reversible*, and the points of any bridge and their images determine an *involution.* The transformation is therefore an *inversion.*

§ 13. *Collineations.*

Assume now that a one-to-one transformation T admits *three invariant pencils* (A), (B), (C). *The indicatrices of the transformation form then a system of conics circumscribed to a fixed triangle ABC.*

Consider a line l of general position, and consider the family of indicatrices relative to the points on this line. They depend on but one parameter and consequently admit of an envelope (E).

Let E be the point in which the conic K relative to a point P touches the envelope. This point is uniquely determined, for by hypothesis *the indicatrix passes through three fixed points.* Let K' be the position of the indicatrix for an infinitely close point P'. The tactal τ of the line l for P is at the intersection of K and l, and likewise that of P' is at the intersection of K' and l. When the point P' approaches P, the point τ approaches the point E as a limit. The point E is therefore on l. Since all indicatrices relative to the line l can not touch the line l it follows therefore that the point E is fixed.

Consequently

Whenever a transformation admits three foci the family of indicatrices relative to the points on any straight line form a pencil of conics.

This theorem leads at once to the following:

Any trifocal transformation is a collineation.

Indeed the image of the line l has all its tangents going through the point E, it is therefore a line through E. We can complete then the preceding theorem by this statement. The four basic points of the pencil of conics are the three foci and the intersection of the line l and its image $\bar{l}$.

Among the indicatrices of the pencil, that relative to E is *tangent to the line* l. For let $\bar{E}$ be the image of E: the line $E\bar{E}$ is the bridge for E, consequently the line l must touch the indicatrix at l. (See Section 1.)

Conversely any collineation admits three foci. To show this I shall prove the following lemma:

The indicatrix of the image of a point P is the image of the indicatrix of the point P.

Indeed, let P and $\bar{P}$ be the two points. By hypothesis *any line l through P is transformed into a line $\bar{l}$ through $\bar{P}$.* Let l and $\bar{l}$ meet in a point τ. This point belongs to the indicatrix K of P. Let us seek the image $\bar{\tau}$ of τ. It is at the intersection of $\bar{l}$ and its image $\bar{\bar{l}}$, and consequently belongs to the indicatrix $\bar{K}$ of $\bar{P}$.

Assume now that the indicatrix K does not degenerate identically, that is that the collineation is *not central.* The two conics K and $\bar{K}$ meet besides in $\bar{P}$ in three other points A, B and C. The line PA goes into $\bar{P}A$, and $\bar{P}A$ into $\bar{\bar{P}}A$. *A is therefore an invariant point* and the same is evidently true of B and C. The three pencils (A), (B), (C) are therefore invariant. Consequently *any collineation is a trifocal transformation, and the indicatrices circumscribe a fixed triangle ABC.*

§ 14. *Perspective.*

In the proof of the last theorem we have expressly assumed that the indicatrix does not degenerate identically. Suppose now that the conic K degenerates at any point of the plane and let P and $\overline{P}$ be a point and its image. We know that the transformation is central, *i. e.*, $P\overline{P}$ goes through a fixed point P. Let β be the second branch of K. Any line l through P will intersect its image $\overline{l}$ in a point τ on β. It follows that if on the line l we take any other point Q the indicatrix of Q will also go through τ. And since τ is any point on β we conclude that

All indicatrices of a central collineation (or perspective) have one branch in common (the axis of perspective).

Any point on the axis β is a focus and the transformation possesses also an additional focus O without the axis. Let C be a point of the axis. The pencil (C) has for the two invariant lines CO and β.

The product of two perspectives is in general a non-central collineation. Indeed, let T_1 and T_2 be two perspectives with O_1 and O_2 as centers, and β_1 and β_2 as axes, respectively, and let P, P_1 and P_2 be a point, its image by T_1, and its image by T_1T_2, respectively. PP_1 goes through O_1, and P_1P_2 through O_2. According to a lemma proved in Section 9 the indicatrix of P by T_1T_2 is determined by P, $\overline{P}$ and the three intersections of the indicatrices of P and P_1 for T_1 and T_2, respectively. If then β_1 and β_2 meet in a point A, PP_1 and β_2 in F, and P_1P_2 and β_1 in G, the indicatrix of the resultant transformation will go through P, P_2, A, F and G, and consequently does not degenerate unless β_1 and β_2 coincide.

The perspectives of a plane do not form a group. All perspectives having a common axis form a group.

It is readily seen that the point A where the two axes of perspective meet is one of the foci of the resultant collineation. To find the other two foci we shall remark that T_1T_2 shifts O_1 and O_2 on the line O_1O_2. The line O_1O_2 is therefore invariant under T_1T_2 and contains the other two foci. The latter can therefore be determined as the intersections of the indicatrix for P and O_1O_2.

Conversely any collineation T can be regarded in an infinite number of ways as a product of two perspectives, T_1T_2. We can choose arbitrarily the two centers O_1 and O_2 on one of the sides of the invariant triangle, the two axes of perspective are then perfectly determined. They meet at the opposite vertex of the triangle. To actually perform the decomposition let P and $\overline{P}$ be two corresponding points and let K be the indicatrix for P. O_1 and O_2 being taken at random on BC, let PO_1 meet K in a second point F, and $\overline{P}O_2$ meet K in G. Then FA and GA are the two axes of perspective. If now P_2 be the intersection of O_1P with $O_2\overline{P}$, P_1 is the image of P by T_1 while $\overline{P}$ is the image of P_1 by T_2. The two perspectives are therefore perfectly determined.

This theorem gives a first method of constructing a collineation given the invariant triangle and a pair of corresponding points.

§ 15. *Classification of Collineations.*

The indicatrix offers a natural method for the *classification* of *collineations.* Since the indicatrices of a collineation form a *bundle of conics* the following cases are possible:

A. *The three foci are distinct.* The indicatrices circumscribe the invariant triangle. We shall call such a collineation *trifocal.* From the standpoint of real transformations we must distinguish two cases: a) the three foci are all *real*, b) two of the foci are *conjugate imaginary.* In the first case all three invariant pencils are of the *hyperbolic* type. In the second case two of the pencils are *elliptic.*

B. *Two of the foci are coincident.* The indicatrices all pass through a fixed point A and touch a fixed line a at another fixed point B. The invariant pencil (B) is *hyperbolic* while (A) is *parabolic.* For any point of the line a the indicatrix degenerates into a and a line β passing through A; while on the other hand the indicatrix of any point on AB will have for the two branches AB and a line through B. In particular at B the indicatrix will consist of AB doubled. Therefore the point A belongs to the discriminant curve. We shall call this type a *bifocal* collineation.

C. All three foci are coincident in a point A. The collineation is *unifocal.* All indicatrices *osculate* at A and therefore have a common tangent a. The pencil (A) is parabolic and the point A again belongs to the discriminant curve.

D. One of the invariant pencils consists of invariant lines only. Suppose (C) to be this pencil; then C is a center and the opposite side c of the invariant triangle is a fixed line. The transformation is therefore a *perspective* having C for *center* and c for *axis.*

E. A particular case of the latter is when the center C is on c. In this case the transformation can be considered as a *degeneracy of the type C.* The curve c is a part of the discriminant curve. The collineation is an *elation.*

§ 16. *The Discriminant of a Collineation.*

The discriminant curve of a trifocal collineation is a parabola inscribed in the invariant triangle.

Consider indeed any line l and its image $\bar{l}$ and let E be their intersection. We have proved that the family of indicatrices relative to l is a pencil of conics having E for a *fourth fundamental point.* But among the conics of a pencil there are two parabolae: let D and D' be the points for which these parabolae are indicatrices. Then D and D' are the unique points in which l meets the discriminant curve Δ. The latter is therefore a conic.

When a point moves along any invariant side, say a, the second branch of its indicatrix turns about A; the correspondence thus generated between the range (a) and the pencil (A) is one-to-one. There exists therefore but one point L on a whose indicatrix degenerates into two parallel lines, *i. e.*, a and the line a' parallel to a and passing through A. This point L belongs to Δ and evidently a touches Δ at L. Since there is but one such point on any side of the invariant triangle, the conic Δ is *inscribed* in the triangle ABC.

On the other hand, the pencil of indicatrices relative to a line l generate a *parabolic* involution on l, since l goes through one of the basic points of the pencil and E is the double point. There exists therefore in a pencil but one conic K touching l, and the point of contact is in E. If we take for l the line at infinity, $\bar{\omega}$, the unique conic of the pencil relative to ω that touches $\bar{\omega}$ is evidently a parabola, and the point of contact belongs to the discriminant curve. The two points D and D' in which $\bar{\omega}$ meets Δ are coincident in E, and therefore D touches $\bar{\omega}$ at E. *The discriminant curve is thus a parabola.*

The asymptotic directions of the indicatrix at a point are the two tangents drawn from P to the discriminant conic Δ.

Indeed, let l be a line through P parallel to its image $\bar{l}$. The point E is then at infinity, and the conics K of the pencil touching l at E are parabolas. The pencil has therefore two coincident parabolas and consequently l is tangent to the discriminant curve Δ.

As a corollary we see that *it is the interior of the parabola Δ that makes up the elliptic region of the plane.* If then the three foci are all real (hyperbolic) the invariant triangle is entirely within the region H. If two of the foci are conjugate imaginary, the real focus, being elliptic, must be within the parabola.

As a second obvious corollary we derive that *the bridges relative to the vanishing line ω envelope the discriminant curve.* It follows from this that the *vanishing line* itself *is a tangent to* Δ, for there exists on it one point V which has for image the point at infinity on ω, and since the image of the bridge of a point touches the indicatrix at the image of the point, V has for indicatrix a parabola and is therefore on Δ.

The same considerations hold with but slight modifications in the case of *bifocal and unifocal* collineation. If A and B are the two foci, and a and c the two invariant lines, the curve will be *inscribed in the angle* (a, c) touching c at A. In the case of a *unifocal* collineation the parabola Δ is osculating at A all indicatrices, it is therefore itself an indicatrix.

In a perspective the conic Δ *degenerates in a double line parallel to the axis of perspective and going through the center. This line coincides with the axis of perspective in the case of elation.*

§ 17. *The Conformal Points of a Collineation.*

Let l and l' be any two lines through a point P, and $\bar{l}$ and $\bar{l}'$ their images through $\bar{P}$. If the angles (l, l') and $(\bar{l}, \bar{l}')$ are equal for all positions of the lines, I shall call P a *conformal point.* I shall prove now that

There exists for any non-conformal collineation one and but one conformal point in the elliptic region. This point is the focus of the discriminating parabola.

Indeed, to determine the point it is sufficient to find the point E which has for indicatrix *the circle circumscribed to the invariant triangle*, and providing the collineation is not conformal, there exists but one such point. Now the asymptotic directions at E are *isotropic*, and since they are, according to the preceding section tangents to Δ, E is the point from which we can draw two isotropic tangents to Δ, that is, the *focus* of Δ.

In the same way it can be shown that in the *hyperbolic* region of the plane there also exists a point H at which angles are conserved. The indicatrix for this point must be an equilateral hyperbola and $H\bar{H}$ a diameter of the indicatrix. Now, since the asymptotic directions of all indicatrices are tangent to Δ, the locus of the point for which the indicatrix is a rectangular hyperbola is the *directrix* d *of the discriminating parabola.* The line d goes through the *orthocenter* of the invariant triangle and meets there its image $\bar{d}$, for all *equilateral hyperbola circumscribed to a triangle pass through its orthocenter. The point* H *may be any point on* d, *but once selected, the collineation is perfectly determined.* Indeed the indicatrix is then known by five points (H, orthocenter, and A, B, C) and consequently by taking for $\bar{H}$ the point diametrically opposed to H on the indicatrix, the indicatrix is determined by the three invariant points and a pair of corresponding points.

The actual construction of the point $\bar{H}$ can be effected in the following manner: Let O be the orthocenter of the triangle. Then OA must be seen from the same angle from H and $\bar{H}$. If then we construct on OA *an arc of capacity* OHA the point $\bar{H}$ is on the arc, providing the arc has been so drawn that the angles $O\bar{H}A$ and OHA have opposite senses of rotation. If we now perform the same construction for OB, the point $\bar{H}$ will be determined as one of the intersections of the two circles, the other being in O.

If a collineation admits more than one conformal point of each type it is a conformal transformation. This is clear for the elliptic points, for were there more than one such point, the corresponding indicatrices being circles, two of the invariant points would be the cyclic points at infinity, and consequently all indicatrices would be circles. As to the hyperbolic conformal points the case can be immediately reduced to the preceding one by a reflection. *A non-central hyperbolic conformal collineation can not exist however, for all conics circumscribing a triangle can not be equilateral hyperbolae.*

§ 18. *Geometrical Determination of a Collineation.*

LEMMA. *If* P, $\bar{P}$ *and* Q, $\bar{Q}$ *are two pairs of corresponding points, and if the indicatrices of the collineation for* P *and* Q *meet in a fourth point* S, *then will* PQ *and* $\bar{P}\bar{Q}$ *meet in* S.

PROBLEM 1. *Determine a collineation by its invariant triangle and two pairs of corresponding points* P, $\bar{P}$ *on* a *and* Q, $\bar{Q}$ *on* b.

The second branch of the indicatrix for P passes through A, and that of Q through B, and according to the lemma both indicatrices contain the point S in which PQ and $\bar{P}\bar{Q}$ meet; the indicatrices are therefore perfectly determined. In order now to obtain the image of any point R in the plane, join R to P and let RP meet SA in R_a, also draw RQ meeting SB in R_b; then will $\bar{P}R_a$ and $\bar{Q}R_b$ meet in the image $\bar{R}$.

PROBLEM 2. *Determine a collineation by its invariant triangle and one pair of corresponding points* R *and* $\bar{R}$ *not belonging to the invariant triangle.*

We can immediately reduce this to Problem 1 by drawing RA, $\bar{R}A$, RB, $\bar{R}B$; for if the first two lines meet a in P and $\bar{P}$, and the second two meet b in Q and $\bar{Q}$, P and $\bar{P}$, Q and $\bar{Q}$ are evidently corresponding points.

The problem can also be treated directly, for the indicatrix K_R of R is determined by the five points R, $\bar{R}$, A, B and C. We can therefore determine projectively the intersection of any line RS with K. Let this intersection be τ; then the indicatrix K_S of S is perfectly determined by the five points τ, S, A, B, C, and the determination of $\bar{S}$ requires the finding of the intersection of $\bar{R}$ with K_S. The construction therefore necessitates two Pascal constructions.*

PROBLEM 3. *Determine a collineation by four pairs of corresponding points of general position:* $P, \bar{P};\ Q, \bar{Q};\ R, \bar{R};\ S, \bar{S}$.

If we denote by $PQ \cdot \bar{P}\bar{Q}$ the intersection of PQ and $\bar{P}\bar{Q}$, the conics K_P and K_Q are determined respectively by the five points

$P;\ \bar{P};\ PQ \cdot \bar{P}\bar{Q};\ PR \cdot \bar{P}\bar{R};\ PS \cdot \bar{P}\bar{S}$ and $Q;\ \bar{Q};\ QP \cdot \bar{Q}\bar{P};\ QR \cdot \bar{Q}\bar{R};\ QS \cdot \bar{Q}\bar{S}$.

By one Pascal construction we can therefore determine the intersection τ_p of K_P with the line MP, M being any point in the plane. A second Pascal construction would give us the point τ_q where MQ meets K_Q. The point $\bar{M}$ is at the intersection of $\bar{P}\tau_p$ and $\bar{Q}\tau_q$.

We have the following theorem as a consequence of the fact that *four pairs of points determine a collineation.*

Let $P, \bar{P};\ Q, \bar{Q};\ R, \bar{R};\ S, \bar{S}$ *be four pairs of points of general position. The four conics* K_P, K_Q, K_R *and* K_S *determined respectively by*

$$P, \bar{P}, PQ \cdot \bar{P}\bar{Q}, PR \cdot \bar{P}\bar{R}, PS \cdot \bar{P}\bar{S};\ Q, \bar{Q}, QP \cdot \bar{Q}\bar{P}, QR \cdot \bar{Q}\bar{R}, QS \cdot \bar{Q}\bar{S};$$
$$R, \bar{R}, RP \cdot \bar{R}\bar{P}, RQ \cdot \bar{R}\bar{Q}, RS \cdot \bar{R}\bar{S};\ S, \bar{S}, SP \cdot \bar{S}\bar{P}, SQ \cdot \bar{S}\bar{Q}, SR \cdot \bar{S}\bar{R}$$

have three points in common.

PROBLEM 4. *A collineation being given by its invariant triangle and a pair of corresponding points,* P *and* $\bar{P}$, *determines its discriminating parabola* Δ *and the elliptic conformal point.*

Applying again our lemma let the parallel to a through A meet the indicatrix K_P in S, then if PS meets a in L, L is the point where a touches Δ. The points M and N are determined in a similar manner.

The *elliptic conformal point* E is located by the following simple construction. Let the parallel through A meet the circumscribing circle K_E in τ, and let τL meet the circle in a second point E. The latter is evidently the *conformal point.*

The latter construction can be applied to the problem: *Find the focus of a parabola inscribed in a triangle.*

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[*] I call Pascal construction the construction based on Pascal's hexagram theorem.

§ 19. *Displacements, Linear Transformations.*

A *conformal* collineation must be either of the *trifocal* type or a *central* collineation.

I. *Rotation.* If the indicatrices of a conformal collineation do not degenerate identically, they can not consist of equilateral hyperbolae only. They are therefore circles, and two of the invariant points are the cyclic points I and J. The transformation is a *rotation. All indicatrices have a common radical center O, the center of rotation.* If the transformation is *real* this latter is *real*, and the *bundle of circles* is a *hyperbolic* bundle. If the center be taken for origin and if α be the angle of rotation, the equation of the transformation is

$$\bar{x}=x\cos\alpha-y\sin\alpha,\quad \bar{y}=x\sin\alpha+y\cos\alpha. \tag{27}$$

The equation of the indicatrix is then

$$X^2+Y^2=k(xX+yY),\ \text{where}\ k=\sec\alpha. \tag{28}$$

II. Still assuming that the *conformity is direct* if the indicatrix degenerates identically, this can only happen in two ways. Either the two branches are *asymptotic* directions of the circle, that is *isotropic* lines and then the transformation is imaginary; or *one of the branches is the line at infinity.* Any line l has in the latter case a *parallel* image. Assume first that the *center of perspective* is not on the *line at infinity.* The transformation is then *homothetic.*

III. If, however, the perspective becomes an *elation*, that is its center is also on the line at infinity, the transformation is a *translation.*

IV. Finally, if the conformity of the collineation is of the *hyperbolic* type, the indicatrix must consist of two perpendicular straight lines, and its *fixed branch must bisect the bridge.* The bridges are therefore all parallel and the *center of perspective is at infinity.* We have then a *reflection.*

More generally consider a collineation leaving the line at infinity invariant. If the two invariant points on the line at infinity are real, all the indicatrices are hyperbolae with *fixed asymptotic directions.* If, on the other hand, the two foci at infinity are imaginary, the transformation is elliptic throughout the plane. The collineation in both cases are *linear transformations*, and, if the finite focus be taken for origin, admit the equations

$$\bar{x}=ax+by,\quad \bar{y}=cx+dy. \tag{29}$$

The discriminant is

$$\Delta=(a-d)^2+4bc, \tag{30}$$

so that the transformation is hyperbolic or elliptic according as Δ is positive or negative. If $\Delta=0$ the two foci at infinity become coincident. The indicatrices represent a bundle of parabolae passing through a fixed point, and admitting at the point a fixed diameter. Geometrically the three cases are distinguished by the fact that a hyperbolic linear transformation will leave two real pencils of parallel lines invariant, a parabolic but one real pencil while an *elliptic linear transformation will not move any real line parallel to itself.*

The last considerations are capable of an obvious generalization. Consider any *Magnus* transformation that admits *two foci at infinity.*

The indicatrices of such a transformation will have fixed asymptotic directions, and, conversely, any one-to-one transformation admitting a system of indicatrices with fixed asymptotic directions is a Magnus transformation with two foci at infinity.

If the third fundamental point be taken for origin and the directions determined by the foci as directions of coordinate axes the equation of such a Magnus transformation takes the form

$$\bar{x}=A/(x-a), \quad \bar{y}=B/(y-b). \tag{31}$$

A and B being arbitrary coefficients, and a, b the coordinates of the finite fundamental point.

§ 20. *Dualistic Developments.*

The results reached in this investigation admit of obvious *dualistic developments.*

Given a *line-to-line transformation* T, let l and $\bar{l}$ be two corresponding lines. Consider a line configuration C admitting l as a simple tangent, and let P be the point of contact. The image $\bar{C}$ of C will touch $\bar{l}$ at a unique point $\bar{P}$. I shall continue to call $P\bar{P}$ the *bridge.* The bridge is independent of the curve C chosen and depends only on the line l and the point P. When the point P describes l there is a *one-to-one correspondence* generated between the ranges l and $\bar{l}$, and the *envelope of the bridge is consequently a conic* L touching l and $\bar{l}$. I shall call *this conic the indicatrix of the line transformation* T *for the line* l.

The discussion of the divers cases brings out the following properties of the *line indicatrix* which I state without proof:

Let S be the intersection of l and $\bar{l}$, and let L touch l and $\bar{l}$ in A and B, respectively, and furthermore let A' and B' be the points of L diametrically opposed to A and B, then

1°. *If a curve C touches l in A its image $\bar{C}$ will touch $\bar{l}$ in S, and conversely: if C touches l in S then will $\bar{C}$ touch $\bar{l}$ in B.*

2°. *If a curve C admits l for asymptotic, its image $\bar{C}$ will touch $\bar{l}$ in A', and if C touches l in B', $\bar{l}$ is an asymptotic of $\bar{C}$.*

3°. *Let C and C' be any two curves and l a common tangent, touching C in P, C' in P'. If $\bar{C}$ and $\bar{C}'$ touch $\bar{l}$ in $\bar{P}$ and $\bar{P}'$, I shall say that distances are conserved along l and $\bar{l}$ if $\overline{PP'}=PP'$.* A line transformation that conserves distances throughout the plane I shall call a *collateral transformation*, directly collateral if $\overline{PP'}=PP'$ inversely if $\overline{PP'}=-PP'$. Then *the condition necessary and sufficient that a transformation be inversely collateral is that the indicatrix be a parabola symmetrically inscribed in the angle $(l, \bar{l})$. For a direct collateral transformation it is necessary and sufficient that the indicatrix degenerate into two points, the point S where l and $\bar{l}$ meet, and the point at infinity in the direction of a bisector of $(l, \bar{l})$.*

4°. If the indicatrix for a line l degenerates, three cases are possible: A. *l is an invariant line;* B. *l is an element of the Jacobian;* C. *Any curve touching l at S will have an image touching $\bar{l}$ at S.* The degenerate indicatrix consists of two points, which in Case A are both on l; in Case B one of the points is on l, the other on $\bar{l}$; finally, in Case C, one of the points is in S.

5°. *If the indicatrix degenerates identically and T is a proper transformation the point S will describe a curve, which may again be called the directrix of T, and T a directed transformation.* If besides T is one-to-one, the directrix is necessarily a *straight line* which I shall call *the axis* of T, and T an *axial transformation.*

6°. If a range of points goes into itself the bearer of the range, f, is a *focal line.* All the indicatrices of the plane will touch f. We can prove in the same way in the correlative case that *a one-to-one transformation will not admit more than three non-concurrent focals.* In the case of three focals we have a collineation which is a *self-dualistic* transformation. If, however, three focals are concurrent in a point O, any line through O is a focal and the transformation is a *perspective.* The latter being an axial transformation, any indicatrix will consist of the point O and a second point on the axis of perspective.

7°. If a collineation T be regarded as a line transformation all *indicatrices are inscribed in the invariant triangle. The discriminating parabola of T regarded as a point collineation is nothing but the indicatrix for the line at infinity of T regarded as a line configuration.*

This outline may suffice to bring out the most prominent features of the analogy between point and line transformations.

§ 21. *Extensions and Generalizations.*

We have expressly assumed in the preceding investigations that our transformation is *of multiplicity one-to-one.* I shall reserve for a subsequent paper the detailed study of *transformations of higher multiplicity.* Nevertheless I think this is the proper place to point out that most of the results reached apply without modifications to the general case.

Let

$$U(\bar{x}, \bar{y}, x, y) = 0, \quad V(\bar{x}, \bar{y}, x, y) = 0, \tag{32}$$

be the equations of the transformation in implicit form, and let $P(x, y)$ be any point in the plane. If, regarding x and y as constants, the solution of equations (32) give a *discrete number of values for* $\bar{x}$ *and* $\bar{y}$ *all distinct* we shall say that the point P is *regular* with respect to the transformation T. If the point is not a regular point it is either a *singular* point, that is, it has an *infinity of corresponding points*, or it is a *point of coincidence*, that is, at least two of its images are coincident.

All our investigations conserving the indicatrix apply evidently to any continuous point to point transformation at any regular point. If the point P has n distinct images $\bar{P}_1, \bar{P}_2, \ldots, \bar{P}_N$, *there will be* n *indicatrices passing through* P, *one for each of its images.* Once the couple of points (x, y) and $(\bar{x}, \bar{y})$ determined, the equation of the indicatrix becomes

$$\begin{vmatrix} \frac{\partial U}{\partial x}(X-x) + \frac{\partial U}{\partial y}(Y-y), & \frac{\partial V}{\partial x}(X-x) + \frac{\partial V}{\partial y}(Y-y), \\ \frac{\partial U}{\partial \bar{x}}(X-\bar{x}) + \frac{\partial U}{\partial \bar{y}}(Y-\bar{y}), & \frac{\partial V}{\partial \bar{x}}(X-\bar{x}) + \frac{\partial V}{\partial \bar{y}}(Y-\bar{y}), \end{vmatrix} = 0. \tag{33}$$

Assume now that the point P *is a point of coincidence.* For example, that T is a two-to-one transformation and the two images of P are coincident in $\bar{P}$. It is clear then that at P the image of any curve C passing through P will have a double point. To a ray t through P will correspond two rays through $\bar{P}$, but to a ray $\bar{t}$, but one ray t will correspond. The indicatrix is therefore cut by any ray through P in one point only. The indicatrix is therefore a cubic having a double point in $\bar{P}$ and passing through P. On the other hand, this cubic is but a *limiting* position of a configuration consisting of two conics, it must therefore degenerate and it consists of *three straight lines* α, β,

γ, of which α and β meet in $\overline{P}$. The point P is evidently on the Jacobian of T. At any other point of the plane the behavior of the indicatrix is regular, and *any determination of* T *can be regarded as a one-to-one transformation.* All the considerations developed in the general theory apply therefore with this restriction.

In the most general case of a continuous p-to-q transformation the behavior of the indicatrix at a point will be analogous to that of the example considered. As long as we avoid the singularities described above, all the results of the general theory hold.

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As to the generalization of the theory, it can be pursued in two directions. First, we may extend the conception of the indicatrix to any *surface of genus zero;* second, to *space line-to-line transformations.* I reserve these considerations for a subsequent communication.

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